AN OBSTRUCTION TO ASYMPTOTIC SEMISTABILITY AND APPROXIMATE CRITICAL METRICS

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1. Introduction

For a polarized algebraic manifold (M, L) with a Kähler metric of constant scalar curvature in the class $c_1(L)_{\mathbb{R}}$, we consider the Kodaira embedding

$$\Phi_{|L^m|} : M \hookrightarrow \mathbb{P}(V_m), \qquad m \gg 1,$$

where $V_m := H^0(M, \mathcal{O}(L^m))^*$. Even when a linear algebraic group of positive dimension acts nontrivially and holomorphically on M, we shall show that the vanishing of an obstruction to asymptotic Chow-semistability allows us to generalize Donaldson's construction [3] of approximate solutions for equations of critical metrics[†] of Zhang [20]. This generalization plays a crucial role in our forthcoming paper [14], in which the asymptotic Chow-stability for (M, L) above will be shown under the vanishing of the obstruction, even when M admits a group action as above.

2. Statement of results

Throughout this paper, we assume that L is an ample holomorphic line bundle over a connected projective algebraic manifold M. Let n and d be respectively the dimension of M and the degree of the image $M_m := \Phi_{|L^m|}(M)$ in the projective space $\mathbb{P}(V_m)$ with $m \gg 1$. Then to this image M_m, we can associate a nonzero element $\hat{M}_m$ of $W_m := \{\mathrm{Sym}^d(V_m)\}^{\otimes n+1}$ such that its natural image $[\hat{M}_m]$ in $\mathbb{P}(W_m)$ is the Chow point associated to the irreducible reduced algebraic cycle M_m on $\mathbb{P}(V_m)$. For the natural action of $H_m := \mathrm{SL}(V_m)$ on W_m and also on $\mathbb{P}(W_m)$, the subvariety M_m of $\mathbb{P}(V_m)$ is said to be *Chow-stable* or *Chow-semistable*, according as the orbit $H_m \cdot \hat{M}$ is closed in W_m or the origin of W_m is not in the closure of $H_m \cdot \hat{M}$ in W_m. Fix an increasing sequence

$$m(1) < m(2) < m(3) < \cdots < m(k) < \cdots \tag{2.1}$$

of positive integers $m(k)$. For this sequence, we say that (M, L) is *asymptotically Chow-stable* or *asymptotically Chow-semistable*, according as for some $k_0 \gg 1$, the subvariety $M_{m(k)}$ of $\mathbb{P}(V_{m(k)})$ is Chow-stable or Chow-semistable for all $k \geq k_0$.

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† In (2.6) below, $\omega = c_1(L; h)$ is called a *critical metric* if $K(q, h)$ is a constant function on M. The same concept was later re-discovered by Luo [12] (see [14]).

Let $\mathrm{Aut}^0(M)$ denote the identity component of the group of all holomorphic automorphisms of M. Then the maximal connected linear algebraic subgroup G of $\mathrm{Aut}^0(M)$ is the identity component of the kernel of the Jacobi homomorphism

$$\alpha_M : \mathrm{Aut}^0(M) \to \mathrm{Aut}^0(\mathrm{Alb}(M)), \quad \text{(cf. [4])}.$$

For the maximal algebraic torus Z in the center of G, we consider the Lie subalgebra $\mathfrak{z}$ of $H^0(M, \mathcal{O}(T^{1,0}M))$ associated to the Lie subgroup Z of $\mathrm{Aut}^0(M)$. For the isotropy subgroup, denoted by $\tilde{S}_m$, of H_m at the point $[\hat{M}_m] \in \mathbb{P}(W_m)$, we have a natural isogeny

$$\iota_m : \tilde{S}_m \to S_m,$$

where S_m is an algebraic subgroup of G. For $Z_m := \iota_m^{-1}(Z)$, we have a Z_m-action on M naturally induced by the Z-action on M. Since the Z-action on M is liftable to a holomorphic bundle action on L (see for instance [7]), the restriction of ι_m to Z_m defines an isogeny of Z_m onto Z. The vector space V_m is viewed as the line bundle $\mathcal{O}_{\mathbb{P}(V_m)}(-1)$ with the zero section blown-down to a point, while the line bundle $\mathcal{O}_{\mathbb{P}(V_m)}(-1)$ coincides with L^{-m} when restricted to M. Hence, the natural $\tilde{S}_m$-action on V_m induces a bundle action of Z_m on L^m which covers the Z_m-action on M. Infinitesimally, each $X \in \mathfrak{z}$ induces a holomorphic vector field $X' \in H^0(L^m, \mathcal{O}(T^{1,0}L^m))$ on L^m. Since the $\mathbb{C}^*$-bundle $L \setminus \{0\}$ associated to L is an m-fold unramified covering of the $\mathbb{C}^*$-bundle $L^m \setminus \{0\}$, the restriction of X' to $L^m \setminus \{0\}$ naturally induces a holomorphic vector field X'' on $L \setminus \{0\}$. Since X'' extends to a holomorphic vector field on L, the mapping $X \mapsto X''$ defines inclusions

$$\rho_m : \mathfrak{z} \hookrightarrow H^0(L, \mathcal{O}(T^{1,0}L)), \qquad m = 1, 2, \dots, \tag{2.2}$$

inducing lifts, from M to L, of vector fields in $\mathfrak{z}$. For a sequence as in (2.1), we say that *the isotropy actions for* (M, L) *are stable* if there exists an integer $k_0 \gg 1$ such that

$$\rho_{m(k)} = \rho_{m(k_0)}, \qquad \text{for all } k \geq k_0. \tag{2.3}$$

For the maximal compact subgroup $(Z_m)_c$ of Z_m, take a $(Z_m)_c$-invariant Hermitian metric λ for L^m. By the theory of equivariant cohomology ([1], [8]), we define (see [15], [13]):

$$\mathcal{C}\{c_1^{n+1}; L^m\}(X) := \frac{\sqrt{-1}}{2\pi}(n+1)\int_M \lambda^{-1}(X\lambda)\, c_1(L^m; \lambda)^n, \quad X \in \mathfrak{z}, \tag{2.4}$$

where $X\lambda$ is as in [13], (1.4.1). Then the $\mathbb{C}$-linear map $\mathcal{C}\{c_1^{n+1}; L^m\} : \mathfrak{z} \to \mathbb{C}$ which sends each $X \in \mathfrak{z}$ to $\mathcal{C}\{c_1^{n+1}; L^m\}(X) \in \mathbb{C}$ is independent of the choice of h. The following gives an obstruction to asymptotic Chow-semistability (see [5], [15], [16] for related results):

Theorem A. *For a sequence as in* (2.1)*, assume that* (M, L) *is asymptotically Chow-semistable. Then for some* $k_0 \gg 1$*, the equality* $\mathcal{C}\{c_1^{n+1}; L^{m(k)}\} = 0$ *holds for all* $k \geq k_0$*. In particular, for this sequence, the isotropy actions for* (M, L) *are stable.*

The following modification of a result in [7] shows that, as an obstruction, the stability condition (2.3) is essential, since the vanishing of (2.4) is straightforward from (2.3).

Theorem B. *For sufficiently large $(n+2)$ distinct integers m_k, $k = 0, 1, \dots, n+1$, suppose that $\rho_{m_0} = \rho_{m_1} = \cdots = \rho_{m_{n+1}}$. Then $\mathcal{C}\{c_1^{n+1}; L^{m_k}\} = 0$ for all k.*

If $\dim Z = 0$, by setting $m(k) = k$ in (2.1) for all $k > 0$, we see that ρ_m are trivial for all $m \gg 1$, and consequently (2.3) holds. Note also that Donaldson's result [3] treating the case $\dim G = 0$ depends on his construction of approximate solutions for equations of critical metrics of Zhang [20]. In Theorem C down below, assuming (2.3), we generalize Donaldson's construction to the case $\dim G > 0$.

Put $N_m := \dim_{\mathbb{C}} V_m - 1$. Let h be a Hermitian metric for L such that $\omega = c_1(L; h)$ is a Kähler metric on M. By the inner product

$$(\sigma, \sigma')_h \; := \int_M < \sigma, \sigma' >_h \; \omega^n, \qquad \sigma, \sigma' \in V_m^*, \tag{2.5}$$

on $V_m^* = H^0(M, \mathcal{O}(L^m))$, we choose a unitary basis $\{\sigma_0^{(m)}, \sigma_1^{(m)}, \dots, \sigma_{N_m}^{(m)}\}$ for V_m^*. Here, $< \sigma, \sigma' >_h$ denotes the function on M obtained as the the pointwise inner product of the sections σ, σ' by the Hermitian metric h^m on L^m. Put

$$K(q, h) := \frac{n!}{m^n} \sum_{i=0}^{N_m} \|\sigma_i^{(m)}\|_h^2, \tag{2.6}$$

where $\|\sigma\|_h^2 :=< \sigma, \sigma >_h$ for all $\sigma \in V_m^*$, and we set $q := 1/m$. We then have the asymptotic expansion of Tian-Zelditch (cf. [18], [19]) for $m \gg 1$:

$$K(q, h) \; = \; 1 + a_1(\omega) q + a_2(\omega) q^2 + a_3(\omega) q^3 + \dots, \tag{2.7}$$

where $a_i(\omega)$, $i = 1, 2, \dots$, are smooth functions on M. Then $a_1(\omega) = \sigma_\omega/2$ (cf. [11]) for the scalar curvature σ_ω of ω. Put $C_q := \{m^n c_1(L)^n[M]/n!\}^{-1}(N_m + 1)$. Then

Theorem C. *For a Kähler metric ω_0 in the class $c_1(L)_{\mathbb{R}}$ of constant scalar curvature, choose a Hermitian metric h_0 for L such that $\omega_0 = c_1(L, h_0)$. For a sequence as in (2.1), assume that the isotropy actions for (M, L) are stable, i.e., (2.3) holds. Put $q = 1/m(k)$. Then there exists a sequence of real-valued smooth functions φ_k, $k = 1, 2, \dots$, on M such that $h(\ell) := h_0 \exp(-\Sigma_{k=1}^{\ell} q^k \varphi_k)$ satisfies $K(q, h(\ell)) - C_q = O(q^{\ell+2})$ for each nonnegative integer ℓ.*

The last equality $K(q, h(\ell)) - C_q = O(q^{\ell+2})$ means that there exist a positive real constant $A = A_\ell$ independent of q such that $\| K(q, h(\ell)) - C_q \|_{C^0(M)} \le A_\ell \, q^{\ell+2}$ for all $0 \le q \le 1$ on M. By [19], for every nonnegative integer j, a choice of a larger constant $A = A_{j,\ell} > 0$ keeps Theorem C still valid even if $C^0(M)$-norm is replaced by $C^j(M)$-norm.

3. An obstruction to asymptotic semistability

The purpose of this section is to prove Theorems A and B. Fix a sequence as in (2.1), and in this section, any kind of stability is considered with respect to this sequence.

Proof of Theorem A: Assume that (M, L) is asymptotically Chow-semistable, i.e., for some $k_0 \gg 1$, the subvariety $M_{m(k)}$ of $\mathbb{P}(V_{m(k)})$ is Chow-semistable for all $k \geq k_0$. Then the isotropy representation of $Z_{m(k)}$ on the line $\mathbb{C} \cdot \hat{M}_{m(k)}$ is trivial (cf. [5], [15]) for $k \geq k_0$, and hence by [15], (3.5) (cf. [16]; [20], (1.5)), we obtain the required equality

$$(3.1) \qquad \mathcal{C}\{c_1^{n+1}; L^{m(k)}\}(X) \;=\; 0, \qquad X \in \mathfrak{z},$$

for all $k \geq k_0$. For λ in (2.4), by setting $h := \lambda^{1/m}$, we have a Hermitian metric h for L. Put $\chi_m := \mathcal{C}\{c_1^{n+1}, L^m\}/m^{n+1}$ for positive integers m. Then by the Leibniz rule,

$$(3.2) \qquad \chi_m(X) \;=\; \frac{\sqrt{-1}}{2\pi}\,(n+1)\int_M h^{-1}(Xh)_{\rho_m}\, c_1(L;h)^n, \qquad X \in \mathfrak{z},$$

where the complexified action $(Xh)_{\rho_m}$ of X on h as in [13], (1.4.1), is taken via the lifting ρ_m in (2.2). Then by (3.1),

$$\chi_{m(k_0)} \;=\; \chi_{m(k_0+1)} \;=\; \cdots\cdots \;=\; \chi_{m(k)} \;=\; \cdots\cdots,$$

and since lifts in (2.2), from M to L, of holomorphic vector fields in $\mathfrak{z}$ are completely characterized by χ_m (cf. [7]), we obtain (2.3), as required. $\square$

Proof of Theorem B: For $q := \text{l.c.m}\{m_k; k = 0, 1, \dots, n+1\}$, we take a q-fold unramified cover $\nu : \tilde{Z} \to Z$ between algebraic tori. Then the Z-action on M naturally induces a $\tilde{Z}$-action on M via this covering. Since ν factors through Z_{m_k}, the lift, from M to L^{m_k}, of the Z_{m_k}-action naturally induces a lift, from M to L^{m_k}, of the $\tilde{Z}$-action. The assumption

$$(3.3) \qquad \rho_{m_0} = \rho_{m_1} = \cdots = \rho_{m_{n+1}}$$

shows that the lifts, from M to L^{m_k}, $k = 0,1,\dots, n+1$, of the $\tilde{Z}$-action come from the same infinitesimal action of $\mathfrak{z}$ as vector fields on L. For brevity, the common ρ_{m_k} in (3.3) will be denoted just by ρ. Then the proof of [6], Theorem 5.1, is valid also in our case, and the formula in the theorem holds. By $Z_{m_k} \subset \mathrm{SL}(V_{m_k})$ and by its contragredient representation, the $\tilde{Z}$-action on $V_{m_k}^* = H^0(M, \mathcal{O}(L^{m_k}))$ comes from an algebraic group homomorphism: $\tilde{Z} \to \mathrm{SL}(V_{m_k}^*)$. Hence, by the notation in (3.2) above, $\int_M h^{-1}(Xh)_\rho\, c_1(L;h)^n = 0$ for all $X \in \mathfrak{z}$, i.e., $\mathcal{C}\{c_1^{n+1}; L^{m_k}\} = 0$ for all k, as required. $\square$

4. Proof of Theorem C

Throughout this section, we assume that the first Chern class $c_1(L)_{\mathbb{R}}$ admits a Kähler metric of constant scalar curvature. Then a result of Lichnérowicz [10] (see also [9]) shows that G is a reductive algebraic group, and consequently the identity component of the center of G coincides with Z in the introduction. Let K be a maximal compact subgroup of G. Then the maximal compact subgroup Z_c of Z satisfies

$$(4.1) \qquad Z_c \;\subset\; K.$$

For an arbitrary K-invariant Kähler metric ω on M in the class $c_1(L)_{\mathbb{R}}$, we write ω as the Chern form $c_1(L;h)$ for some Hermitian metric h for L. Let $\Psi(q,\omega)$ denote the power series in q given by the right-hand side of (2.7). Then

$$\int_M \{\Psi(q,\omega) - C_q\}\,\omega^n \;=\; \int_M \left\{ -C_q + \frac{n!}{m^n}\sum_{i=0}^{N_m} \|\sigma_i^{(m)}\|_h^2 \right\} \omega^n \;=\; 0. \tag{4.2}$$

Let h_0 be a Hermitian metric for L such that $\omega_0 := c_1(L;h_0)$ is a Kähler metric of constant scalar curvature on M. We write

$$\omega_0 \;=\; \frac{\sqrt{-1}}{2\pi}\sum_{\alpha,\beta} g_{\alpha\bar\beta}dz^\alpha\wedge dz^{\bar\beta},$$

for a system $(z^1, z^2, \dots, z^n)$ of holomorphic local coordinates on M. In view of [10] (see also [9]), replacing ω_0 by $g^*\omega_0$ for some $g \in G$ if necessary, we may assume that ω_0 is K-invariant. Let D_0 be the Lichnérowicz operator, as defined in [2], (2.1), for the Kähler manifold (M,ω_0). Since ω_0 has a constant scalar curvature, D_0 is a real operator. Let $\mathcal{F}$ denote the space of all real-valued smooth K-invariant functions φ such that $\int_M \varphi\omega_0^n = 0$. Since the operator D_0 preserves the space $\mathcal{F}$, we write D_0 as an operator $D_0 : \mathcal{F} \to \mathcal{F}$, and the kernel in $\mathcal{F}$ of this operator will be denoted by $\operatorname{Ker} D_0$. Let $\mathfrak{z}_c$ denote the Lie subalgebra of $\mathfrak{z}$ corresponding to the maximal compact subgroup Z_c of Z. Then

$$\gamma : \operatorname{Ker} D_0 \;\cong\; \mathfrak{z}_c, \qquad \eta \leftrightarrow \gamma(\eta) := \operatorname{grad}^{\mathbb{C}}_{\omega_0}\eta, \tag{4.3}$$

where $\operatorname{grad}^{\mathbb{C}}_{\omega_0}\eta := (1/\sqrt{-1})\Sigma\, g^{\bar\beta\alpha}\eta_{\bar\beta}\partial/\partial z^\alpha$ denotes the complex gradient of η with respect to ω_0. We then consider the orthogonal projection

$$P \;:\; \mathcal{F}\ (= \operatorname{Ker} D_0 \oplus \operatorname{Ker} D_0^\perp) \;\to\; \operatorname{Ker} D_0.$$

Starting from $h(0) = h_0$ and $\omega(0) := \omega_0$, we inductively define a Hermitian metric $h(k)$ for L, and a Kähler metric $\omega(k) := c_1(L;h(k))$, called the *k-approximate solution*, by

$$h(k) \;=\; h(k-1)\,\exp(-q^k\varphi_k), \qquad\qquad k = 1,2,\dots,$$

$$\omega(k) \;=\; \omega(k-1) + \frac{\sqrt{-1}}{2\pi}\,q^k\,\partial\bar\partial\varphi_k, \qquad k = 1,2,\dots,$$

for a suitable function $\varphi_k \in \operatorname{Ker} D_0^\perp$, where we require $h(k)$ to satisfy $K(q,h(k)) - C_q = O(q^{k+2})$. In other words, by (4.2), each $\omega(k)$ is required to satisfy the following conditions:

$$(1-P)\{\Psi(q,\omega(k)) - C_q\} \;\equiv\; 0, \quad \text{modulo } q^{k+2}, \tag{4.4}$$

$$P\{\Psi(q,\omega(k)) - C_q\} \;\equiv\; 0, \quad \text{modulo } q^{k+2}. \tag{4.5}$$

If $k = 0$, then $\omega(0) = \omega_0$, and by [11], both (4.4) and (4.5) hold for $k = 0$. Hence, let $\ell \geq 1$ and assume (4.4) and (4.5) for $k = \ell - 1$. It then suffices to find $\varphi_\ell \in \operatorname{Ker} D_0^\perp$ satisfying both (4.4) and (4.5) for $k = \ell$. Put

$$\Phi(q,\varphi) \;:=\; (1-P)\left\{\,\Psi\left(q,\,\omega(\ell-1) + (\sqrt{-1}/2\pi)\,q^\ell\,\partial\bar\partial\varphi\right) - C_q\right\}, \qquad \varphi \in \operatorname{Ker} D_0^\perp.$$

Then by (4.4) applied to $k = \ell - 1$, we have $\Phi(q,0) \equiv u_\ell \, q^{\ell+1}$ modulo $q^{\ell+2}$, where u_ℓ is a function in $\operatorname{Ker} D_0^\perp$. Since $2\pi\omega(\ell-1) = 2\pi\omega_0 + \sqrt{-1}\,\Sigma_{k=1}^{\ell-1} q^k \partial\bar\partial\varphi_k$, we have $\omega(\ell-1) = \omega_0$ at $q = 0$. Since the scalar curvature of ω_0 is constant, the variation formula for the scalar curvature (see for instance [2], (2.5); [3]) shows that

$$\Phi(q,\varphi_\ell) \;\equiv\; \Phi(q,0) - q^{\ell+1}(D_0\varphi_\ell/2) \;\equiv\; (2u_\ell - D_0\varphi_\ell)\,(q^{\ell+1}/2),$$

modulo $q^{\ell+2}$. Since u_ℓ is in $\operatorname{Ker} D_0^\perp$, there exists a unique $\varphi_\ell \in \operatorname{Ker} D_0^\perp$ such that $2u_\ell = D_0\varphi_\ell$ on M. Fixing such φ_ℓ, we obtain $h(\ell)$ and $\omega(\ell)$. Thus (4.4) is true for $k = \ell$.

Now, we have only to show that (4.5) is true for $k = \ell$. Before checking this, we give some preliminary remarks. Note that $C_q = 1 + O(q)$. Moreover, by (2.7), $\Psi(q,\omega) = 1 + q\{a_1(\omega) + a_2(\omega)q + \dots\}$, and hence

$$\begin{aligned}\Psi(q,\omega(\ell)) - C_q \;&=\; \Psi\left(q,\, \omega(\ell-1) + (\sqrt{-1}/2\pi)\, q^\ell \partial\bar\partial\varphi_\ell\right) - C_q\\ &\equiv\; \Psi(q,\omega(\ell-1)) - C_q \;\equiv\; 0, \qquad\quad \text{modulo } q^{\ell+1}.\end{aligned}$$

By [17], p. 35, the G-action on M is liftable to a bundle action of G on the real line bundle $(L\cdot\bar L)^{1/2} = (L^m\cdot\bar L^m)^{1/2m}$. Then the induced K-action on $(L\cdot\bar L)^{1/2}$ is unique, because liftings, from M to L^m, of the G-action differ only by scalar multiplications of L^m by characters of Z. In this sense, $h(\ell)$ is K-invariant. Put $r := \dim_{\mathbb{C}} Z$. Then we can write $Z_m \;=\; \mathbb{G}_m^r \;=\; \{\, t = (t_1, t_2, \dots, t_r) \in (\mathbb{C}^*)^r \,\}$. By the natural inclusion

$$\psi_m \;:\; Z_m \;\hookrightarrow\; H_m = \mathrm{SL}(V_m),$$

we can choose a unitary basis $\{\tau_0, \tau_1, \dots, \tau_{N_m}\}$ for $(V_m^*, (\ ,\ )_{h(\ell)})$ (cf. (2.5)) such that, for some integers α_{ij} with $\Sigma_i\,\alpha_{ij} = 0$, the contragredient representation ψ_m^* of ψ_m is given by

$$\psi_m^*(t)\,\tau_i \;=\; \left(\prod_{j=1}^{r} t_j^{\alpha_{ij}}\right)\tau_i, \qquad i = 0, 1, \dots, N_m,$$

for all $t \in (\mathbb{C}^*)^r = Z_m$. Now by (2.3), for some $\rho : \mathfrak{z} \hookrightarrow H^0(L, \mathcal{O}(T^{1,0}L))$, we can write $\rho_{m(k)} = \rho$ for all $k \geq k_0$. Consider the Kähler metric $\omega_m := c_1(L; h_m)$ on M in the clasas $c_1(L)_{\mathbb{R}}$, where $h_m := (|\tau_0|^2 + |\tau_1|^2 + \dots + |\tau_{N_m}|^2)^{-1/m}$. From now on, let $m = m(k)$, where k is running through all integers $\geq k_0$. Put $X_j := t_j\partial/\partial t_j$. Then $\{X_1, X_2, \dots, X_r\}$ forms a $\mathbb{C}$-basis for the Lie algebra $\mathfrak{z}$ such that, using the notation as in (3.2), we have

$$(4.6)\qquad h_m^{-1}(X_j h_m)_\rho \;=\; -\frac{\Sigma_i\,\alpha_{ij}|\tau_i|^2}{m\,\Sigma_i\,|\tau_i|^2}, \quad 1 \leq j \leq r, \quad \text{for } m = m(k) \text{ with } k \geq k_0,$$

where in the numerator and the denominator, the sum is taken over all integers i such that $0 \leq i \leq N_m$. From (2.3) and Theorem B, using the notation as in (3.2), we obtain

$$(4.7)\qquad \int_M h(\ell)^{-1}(X_j h(\ell))_\rho\, \omega(\ell)^n \;=\; 0, \qquad 1 \leq j \leq r.$$

By $\int_M h_0^{-1}(X_j h_0)_\rho\,\omega_0^n / \int_M \omega_0^n = 0$, we have $\eta_j := h_0^{-1}(X_j h_0)_\rho \in \operatorname{Ker} D_0$. Then $\gamma(\eta_j) = \sqrt{-1}\,X_j$. Hence $\{\eta_1, \eta_2, \dots, \eta_r\}$ is an $\mathbb{R}$-basis for $\operatorname{Ker} D_0$. Since $\Psi(q, \omega(\ell)) \equiv C_q$ modulo $q^{\ell+1}$, it follows that

$$-C_q + \frac{n!}{m^n}\sum_{i=0}^{N_m} \|\tau_i\|^2_{h(\ell)} \;\equiv\; v_\ell\, q^{\ell+1} \tag{4.8}$$

modulo $q^{\ell+2}$ for some $v_\ell \in \operatorname{Ker} D_0$, because (4.4) is true for $k = \ell$. In view of (4.2), (4.6), $h_m - h_0 = O(q)$ and $\omega(\ell) - \omega_0 = O(q)$, we see from (4.8) that, modulo $q^{\ell+2}$,

$$\begin{aligned}
q^{\ell+1}\int_M \eta_j\, v_\ell\, \omega_0^n \;&\equiv\; \int_M \eta_j \left(-C_q + \frac{n!}{m^n}\sum_{i=0}^{N_m}\|\tau_i\|^2_{h(\ell)}\right)\{\omega(\ell)\}^n\\
&\equiv\; \int_M h_0^{-1}(X_j h_0)_\rho \left(-C_q + \frac{n!}{m^n}\sum_{i=0}^{N_m}\|\tau_i\|^2_{h(\ell)}\right)\{\omega(\ell)\}^n\\
&\equiv\; \int_M h_m^{-1}(X_j h_m)_\rho \left(-C_q + \frac{n!}{m^n}\sum_{i=0}^{N_m}\|\tau_i\|^2_{h(\ell)}\right)\{\omega(\ell)\}^n\\
&\equiv\; \int_M \frac{\Sigma_i\,\alpha_{ij}\|\tau_i\|^2_{h(\ell)}}{m\,\Sigma_i\,\|\tau_i\|^2_{h(\ell)}} \left(C_q - \frac{n!}{m^n}\sum_{i=0}^{N_m}\|\tau_i\|^2_{h(\ell)}\right)\{\omega(\ell)\}^n,
\end{aligned}$$

Since $\Sigma_i \alpha_{ij} = 0$ for all j, we obtain, modulo $q^{\ell+2}$,

$$\begin{aligned}
q^{\ell+1}\int_M \eta_j\, v_\ell\, \omega_0^n \;&\equiv\; C_q\int_M \frac{\Sigma_i\,\alpha_{ij}\|\tau_i\|^2_{h(\ell)}}{m\,\Sigma_i\,\|\tau_i\|^2_{h(\ell)}}\{\omega(\ell)\}^n \;\equiv\; C_q\int_M h_m^{-1}(X_j h_m)_\rho\{\omega(\ell)\}^n\\
&\equiv\; C_q\int_M \left\{h_m^{-1}(X_j h_m)_\rho - h(\ell)^{-1}(X_j h(\ell))_\rho\right\}\{\omega(\ell)\}^n,
\end{aligned}$$

where the equivalence just above follows from (4.7). The last integrand is rewritten as

$$\begin{aligned}
&h_m^{-1}(X_j h_m)_\rho - h(\ell)^{-1}(X_j h(\ell))_\rho \;=\; X_j \log\{h_m/h(\ell)\} \;=\; -\,\frac{1}{m}\,X_j \log\left(\frac{n!}{m^n}\sum_{i=0}^{N_m}\|\tau_i\|^2_{h(\ell)}\right)\\
&\equiv\; -\,q\,X_j\log(C_q + v_\ell\, q^{\ell+1}) \;\equiv\; -\,C_q^{-1}(X_j v_\ell) q^{\ell+2} \;\equiv\; 0, \qquad \bmod q^{\ell+2}.
\end{aligned}$$

Therefore, $\int_M \eta_j v_\ell\, \omega_0^n = 0$ for all j. From $v_\ell \in \operatorname{Ker} D_0$, it now follows that $v_\ell = 0$. This shows that (4.5) is true for $k = \ell$, as required. □

5. Concludung remarks

As in Donaldson's work [3], the construction of approximate solutions in Threorem C is a crucial step to the approach of the stability problem for a polarized algebraic manifold with a Kähler metric of constant scalar curvature. Actually, in a forthcoming paper [14], this construction allows us to prove the following:

Theorem. *For a sequence as in* (2.1), *assume that the isotropy actions for* (M, L) *are stable. Assume further that* $c_1(L)_{\mathbb{R}}$ *admits a Kähler metric of constant scalar curvature. Then for this sequence,* (M, L) *is asymptotically Chow-stable.*

Moreover, if a sequence (2.1) exists in such a way that (2.3) holds, then the same argument as in the case $\dim G = 0$ (cf. [3]) is applied, and we can also show the uniquness, modulo the action of G, of the Kähler metrics of constant scalar curvature in the polarization class $c_1(L)_{\mathbb{R}}$. We finally remark that, if $\dim G = 0$, the asymptotic Chow-stability implies the asymptotic stability in the sense of Hilbert schemes (cf. [17], p.215). Hence the result of Donaldson [3] follows from the theorem just above.

References

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